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1988 J. Phys. A: Math. Gen. 21 593

(http://iopscience.iop.org/0305-4470/21/3/015)

J. Phys. A: Math. Gen. **21** (1988) 593-597. Printed in the UK

A six-variable identity for a ternary vector cross product in eight-dimensional space

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Received 1 September 1987

Abstract. Let $X(a, b, c)$ be a ternary vector cross product for eight-dimensional Euclidean space E. An identity is derived which expresses $\langle X(a, b, c), X(u, v, w)\rangle$ in terms of the Spin(7)-invariant scalar quadruple product $\Phi(a, b, c, d) = \langle a, X(b, c, d)\rangle$. The proof of the identity is coordinate free, and starts out from an explicit expression for X, with E viewed as complex four-dimensional Hilbert space.

Let $(E, \langle\ ,\ \rangle, X)$ be an rXn algebra, $2 \leq r \leq n$, as defined in [1]. Spelling this out, E is a real n-dimensional vector space which is equipped with a positive-definite inner product $\langle\ ,\ \rangle$ and also with an r-fold vector cross product, that is with a map $X: E^r \to E$ which satisfies the axioms (cf [2, 3]):

$$X \text{ is } r\text{-linear} \tag{1}$$

$$\langle X(a_1, \ldots, a_r), a_i\rangle = 0 \qquad i = 1, 2, \ldots, r \tag{2}$$

$$\langle X(a_1, \ldots, a_r), X(a_1, \ldots, a_r)\rangle = \langle a_1 \wedge \ldots \wedge a_r | a_1 \wedge \ldots \wedge a_r\rangle \tag{3}$$

where $\langle a_1 \wedge \ldots \wedge a_r | b_1 \wedge \ldots \wedge b_r\rangle \equiv \det(\langle a_i, b_j\rangle)$. Associated with an rXn algebra is the 'scalar $(r+1)$-tuple product' Φ defined by

$$\Phi(a_0, a_1, \ldots, a_r) = \langle a_0, X(a_1, \ldots, a_r)\rangle. \tag{4}$$

By axioms (1) and (2), Φ is alternating, whence so is X. Consequently we may view X as a linear map $\bigwedge^r E \to E$.

It is known ([2, 3], see also [1]) that $2Xn$ algebras exist only in dimensions $n = 3, 7$, that $3Xn$ algebras exist only in dimensions $n = 4, 8$, and that for $r > 3$, rXn algebras exist only in dimension $r+1$. In the case of the 'non-exceptional' $rX(r+1)$ algebras it is easy to see that axioms (1)-(3) possess precisely two solutions for each $r \geq 2$, given in terms of the star operator $\bigwedge^r E \to E$ by

$$X(a_1, \ldots, a_r) = \pm{}^*(a_1 \wedge \ldots \wedge a_r). \tag{5}$$

Now, for Euclidean space E, the star operator is well known to be an isometry. Consequently in these non-exceptional cases the following strengthened form of (3) holds:

$$\langle X(a_1, \ldots, a_r), X(b_1, \ldots b_r)\rangle = \langle a_1 \wedge \ldots \wedge a_r | b_1 \wedge \ldots \wedge b_r\rangle. \tag{6}$$

Going in the other direction it should be noted that if we had adopted (1), (2) and (6) as axioms, instead of (1)-(3), then we would have overlooked the exceptional $2X7$

and $3X8$ cases, which are in fact of greatest interest! For, viewing X as $\bigwedge^r E \to E$, we see that (6) implies that X is an isometry, and hence that $\dim(\bigwedge^r E) = \dim E = n$, which (for $r \geqslant 2$) occurs only in the non-exceptional cases $r = n-1$. The purpose of this paper is to obtain the generalisations of (6) which are valid in the exceptional $2X7$ and $3X8$ cases, namely those given in the following two theorems. In the case of a $2X7$ algebra we will denote the underlying seven-dimensional Euclidean space by E' rather than E, and will denote the scalar triple product by ϕ, not Φ:

$$\phi(a, b, c) = \langle a, X(b, c)\rangle. \tag{7}$$

Since ϕ is alternating we think of it also as an element $\phi \in \bigwedge^3 E'$ and denote by $*$ a (suitably signed) star operator $\bigwedge^3 E' \to \bigwedge^4 E'$.

Theorem A. For a $2X7$ algebra $(E', \langle\ ,\ \rangle, X)$ we have the identity

$$\langle X(a, b), X(u, v)\rangle = \langle a \wedge b \mid u \wedge v\rangle + \psi(a, b, u, v) \tag{8}$$

where $\psi = {*\phi}$.

Theorem B. For a $3X8$ algebra $(E, \langle\ ,\ \rangle, X)$ we have

$$\langle X(a, b, c), X((u, v, w)\rangle = \langle a \wedge b \wedge c \mid u \wedge v \wedge w\rangle + F(a, b, c, u, v, w) \tag{9}$$

where (writing $f(a, b, c) + f(b, c, a) + f(c, a, b)$ as $\mathrm{Cyc}_{a,b,c} f(a, b, c)$)

$$F(a, b, c, u, v, w) = \varepsilon\, \mathrm{Cyc}_{a,b,c}\, \mathrm{Cyc}_{u,v,w} \langle a, u\rangle \Phi(b, c, v, w) \tag{10}$$

with $\varepsilon = +1$ or -1 according to whether the $3X8$ algebra is of type I or type II (see [1]).

Remark. Some slight insight into the identity (8) can be gained from the following alternative proof of (6) in the case of the familiar vector cross product $\boldsymbol{a} \times \boldsymbol{b}$ in three-dimensional Euclidean space. In this $2X3$ case let us *define* ψ by

$$\psi(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{u} \times \boldsymbol{v}\rangle - \langle \boldsymbol{a} \wedge \boldsymbol{b} \mid \boldsymbol{u} \wedge \boldsymbol{v}\rangle. \tag{11}$$

By axiom (3) we have

$$\psi(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a}, \boldsymbol{b}) = 0. \tag{12}$$

(From the point of view of the teaching of $\boldsymbol{a} \times \boldsymbol{b}$ to undergraduates via the geometric definition $\boldsymbol{a} \times \boldsymbol{b} = \|\boldsymbol{a}\|\,\|\boldsymbol{b}\| \sin\theta\, \boldsymbol{n}$, then (12) holds because $\sin^2\theta = 1 - \cos^2\theta$.) Observing that $\psi(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a}, \boldsymbol{v}) = \psi(\boldsymbol{a}, \boldsymbol{v}, \boldsymbol{a}, \boldsymbol{b})$, linearisation of (12) in the vector $\boldsymbol{b}$ yields

$$\psi(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a}, \boldsymbol{v}) = 0. \tag{13}$$

Consequently $\psi(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{u}, \boldsymbol{v})$, which (trivially) is zero when $\boldsymbol{a} = \boldsymbol{b}$ and when $\boldsymbol{u} = \boldsymbol{v}$, is also zero when $\boldsymbol{a} = \boldsymbol{u}$. The quadrilinear form ψ is thus alternating and hence, in dimension 3, is the zero form. This proof of (6) (and hence of the familiar identity $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{u} = \langle \boldsymbol{a}, \boldsymbol{u}\rangle \boldsymbol{b} - \langle \boldsymbol{b}, \boldsymbol{u}\rangle \boldsymbol{a}$) fails for dimension $n > 3$, where ψ is still alternating but may not be zero. So we are led to speculate, could ψ, for $n > 3$, be simply related to ϕ? Conceivably we could be led in this way to consideration of the exceptional $2X7$ algebras, since only in dimension 7 $(=4+3)$ can ψ and ϕ, considered as elements of $\bigwedge^4 E'$ and $\bigwedge^3 E'$, be related by the star operator.

In terms of the tensor components $\Phi_{abcd} = \Phi(e_a, e_b, e_c, e_d)$ of Φ relative to an orthonormal basis, the identity (10) takes on the form

$$\Phi^{abcs}\Phi_{uvws} = 6\delta^{abc}_{uvw} + 9\varepsilon\delta^{[a}_{[u}\Phi^{bc]}{}_{vw]}. \tag{14}$$

In this coordinate form the identity (10) has in fact previously appeared in the physics literature on $d = 11$ supergravity theories (see [4, 5]). The tensor Φ_{abcd} is totally skew-symmetric, self-dual (as noted also in [1]) and invariant under the Spin(7) subgroup of $SO(E) \simeq SO(8)$ which is the automorphism group of the $3X8$ algebra. (Conceivably, as discussed in [6], the invariance group of Φ could even be bigger than Spin (7).)

In [4] the eight-dimensional identity (14) was obtained by making use of a seven-dimensional identity (the coordinate form of (8)). In the present paper we instead straightaway pursue a coordinate-free proof of identity (9). The identity (8) can then be obtained from (9) as follows.

Choose any unit vector $e \in E$ and let E' denote the seven-dimensional subspace which is orthogonal to e. Then E' becomes a $2X7$ algebra upon defining

$$X(a, b) = X(a, e, b) \qquad \text{for } a, b \in E'. \tag{15}$$

The associated scalar triple product $\phi(a, b, c) = \langle a, X(b, c)\rangle$ is thus related to Φ by

$$\phi(a, b, c) = \Phi(e, a, b, c) \qquad a, b, c \in E'.$$

Consequently, after using the self-duality of Φ, the special case $c = w = e$ and $a, b, u, v \in E'$ of identity (9) is seen to yield identity (8).

Our proof of theorem B will start out from certain explicit expressions for X and Φ which we now describe. Let us find E as the realisation $E = (C^4)^{\mathbb{R}}$ of complex four-dimensional Hilbert space C^4. We denote the inner product on C^4 by (a, b) and take it to be linear in a and so antilinear in b. Let $\mathbf{\Delta}$ denote a determinant function for C^4, normalised to be equal to $+1$ upon some ordered orthonormal basis $\{e_0, e_1, e_2, e_3\}$ for C^4. Let $b \times c \times d$ denote the 'complex ternary vector cross product' on C^4 which is defined by

$$\Delta(a, b, c, d) = (a, b \times c \times d). \tag{16}$$

This cross product satisfies a peculiar kind of complex version of properties (1), (2) and (6):

$$a_1 \times a_2 \times a_3 \text{ is triantilinear in } a_1, a_2, a_3 \tag{17}$$

$$(a_1 \times a_2 \times a_3, a_i) = 0 \qquad i = 1, 2, 3 \tag{18}$$

$$(a_1 \times a_2 \times a_3, b_1 \times b_2 \times b_3) = \det(b_i, a_j). \tag{19}$$

Finally, let $\langle a, b\rangle$ and $[a, b]$ denote the real and imaginary parts of (a, b):

$$(a, b) = \langle a, b\rangle + \mathrm{i}[a, b]. \tag{20}$$

Thus E is equipped now not only with O(8) geometry by means of $\langle\ ,\ \rangle$, but also with $Sp(8; \mathbb{R})$ geometry by means of $[\ ,\ ]$.

We now claim that a ternary vector cross product X for $(E, \langle\ ,\ \rangle)$ is given by

$$X(a, b, c) = a \times b \times c + \mathrm{i}\, \mathrm{Cyc}_{a,b,c}[a, b]c \tag{21}$$

the associated scalar quadruple product being therefore

$$\Phi(a, b, c, d) = \mathrm{Re}(\Delta(a, b, c, d)) + \mathrm{Cyc}_{a,b,c}[a, b][c, d]. \tag{22}$$

It is possible to check directly that X so defined does satisfy the axioms (1)-(3). However, another method of carrying out this check is slightly cleaner. Define a map $\{\ \}: E^3 \to E$ by (cf [1])

$$\{abc\} = X(a, b, c) + \langle a, b\rangle c + \langle b, c\rangle a - \langle a, c\rangle b \tag{23}$$

i.e. in our present case, by

$$\{abc\} = a \times b \times c + (a, b)c + (b, c)a - (a, c)b. \tag{24}$$

Then one can check that $\{\ \}$ as given by (24) enjoys the properties

$$\{\ \} \text{ is trilinear} \tag{1'}$$

$$\{aac\} = \langle a, a\rangle c = \{caa\} \tag{2'}$$

$$\langle \{abc\}, \{abc\}\rangle = \langle a, a\rangle\langle b, b\rangle\langle c, c\rangle. \tag{3'}$$

As demonstrated in [6, theorem 2.3], checking that $\{\ \}$ satisfies (1′)-(3′) is completely equivalent to proving that X satisfies (1)-(3). As noted also in [6], X is in fact of type I.

The way ahead is now clear: definition (21) combined with property (19) will surely allow us to 'evaluate' the inner product $\langle X(a, b, c), X(u, v, w)\rangle$. However, in order to simplify the result to the desired form as given in theorem B, the following preliminary lemma is helpful.

Lemma. The following identity holds:

$$\mathrm{Cyc}_{a,b,c}\, \mathrm{Cyc}_{u,v,w} \langle a, u\rangle\, \mathrm{Re}(\Delta(b, c, v, w))$$
$$= \mathrm{Cyc}_{u,v,w}[u, v]\, \mathrm{Im}(\Delta(a, b, c, w)) + \mathrm{Cyc}_{a,b,c}[a, b]\, \mathrm{Im}(\Delta(u, v, w, c)).$$

Proof of lemma. Start from the fact ('Cramer's rule') that alternation of the *quinque*linear function $\Delta(a_1, a_2, a_3, a_4)a_5$ of four-dimensional complex vectors $a_1, \ldots, a_5$ must yield zero. Upon forming the inner product of this result with a sixth vector, and changing the notation, we have

$$\mathrm{Cyc}_{a,b,c}(a, u)\Delta(b, c, v, w) = (v, u)\Delta(a, b, c, w) - (w, u)\Delta(a, b, c, v). \tag{25}$$

Equally well we also have

$$\mathrm{Cyc}_{u,v,w}(u, a)\Delta(b, c, v, w) = (b, a)\Delta(u, v, w, c) - (c, a)\Delta(u, v, w, b). \tag{26}$$

By adding $\mathrm{Cyc}_{u,v,w}$ (25) to $\mathrm{Cyc}_{a,b,c}$ (26) and taking the real part we obtain the result announced in the lemma.

Proof of theorem B.

$$\langle X(a, b, c), X(u, v, w)\rangle$$
$$= \mathrm{Re}(X(a, b, c), X(u, v, w))$$
$$= \mathrm{Re}(D) + \mathrm{Cyc}_{u,v,w}[u, v]\, \mathrm{Im}(\Delta(a, b, c, w) + \mathrm{Cyc}_{a,b,c}[a, b]\, \mathrm{Im}(\Delta(u, v, w, c))$$
$$+ \mathrm{Cyc}_{a,b,c}\, \mathrm{Cyc}_{u,v,w}[a, b][u, v]\langle c, w\rangle \tag{27}$$

(since $\mathrm{Re}(a \times b \times c, iu) = \mathrm{Im}(\Delta(a, b, c, u))$, etc), where

$$D \equiv \begin{vmatrix} (a, u) & (a, v) & (a, w) \\ (b, u) & (b, v) & (b, w) \\ (c, u) & (c, v) & (c, w) \end{vmatrix}.$$

The real part of the 3×3 complex determinant D contributes $\langle a \wedge b \wedge c | u \wedge v \wedge w \rangle$ plus other terms. These latter, taken along with the fourth term on the RHS of (27), contribute an amount

$$[b, u][a, v] + [u, a][b, v] + [a, b][u, v]$$

to the coefficient of $\langle c, w \rangle$, i.e. by (22) an amount

$$\Phi(a, b, u, v) - \mathrm{Re}(\Delta(a, b, u, v)).$$

By virtue of the lemma the terms involving $\mathrm{Re}(\Delta)$ and $\mathrm{Im}(\Delta)$ cancel, and theorem B ensues, with $\varepsilon = +1$. Replacing $X \in$ type I with $-X \in$ type II gives theorem B in the case $\varepsilon = -1$.

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